Phase Transitions on Fixed Connected Graphs and Random Graphs in the Presence of Noise

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Abstract

In this paper, we study phase transition behavior emerging from the interactions among multiple agents in the presence of noise. We propose a simple discrete-time model in which a group of non-mobile agents form either a fixed connected graph or a random graph process, and each agent, taking bipolar value either $+1$ or -1, updates its value according to its previous value and the noisy measurements of the connected agents' values. We present proofs for the occurrence of the following phase transition behavior: At a noise level higher than some threshold, the system generates symmetric behavior (vapor or melt of magnetization) or disagreement; whereas at a noise level lower than the threshold, the system exhibits spontaneous symmetry breaking (solid or magnetization) or consensus. The threshold is found analytically. The phase transition holds for any dimension. Finally, we demonstrate the phase transition behavior and all analytic results using simulations. This result may be found useful in the study of the collective behavior of complex systems under communication constraints.

I. Introduction

Phase transition in a system refers to the sudden change of a system property as some parameter of the system crosses a certain threshold value. Phase transitions have been observed in a wide variety of studies, such as in physics, chemistry, biology, complex systems, computer science, and random graphs, to list a few. It leads to long term attention in the literature, from physicists such as Ising [1] in the 1920's to mathematicians such as Erdös and Rényi [2] in the 1960's, from complex systems theorists such as Langton [3] in the 1990's to control scientists such as Olfati-Saber [4] in the 2000's.

Ising and other physicists have thoroughly studied the simple but "realistic enough" Ising model, for the understanding of phase transitions in magnetism, lattice gases, etc. In an Ising model, each node can have two values, and the neighboring nodes have an energetic preference

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to take the same value, under some constraints such as a temperature one. It is observed that, for an Ising model with dimension at least 2, a temperature higher than a critical point leads to symmetric behavior (e.g., “melt” of magnetization, or vapor), whereas a temperature lower than that point leads to asymmetric behavior (e.g., magnetization, or liquid). The Ising model is a discrete-time discrete-state model, and is closely related to the Hopfield networks and cellular automata.

Erdös and Rényi [2] showed that, graphs of sizes slightly less than a certain threshold are very unlikely to have some properties, whereas graphs with a few more edges are almost certain to have these properties. This is called phase transition of random graphs, see for example [5].

Viscek *et al* [6] showed that a two-dimensional nonlinear model exhibits a phase transition in the sense of spontaneous symmetry breaking as the noise level crosses a threshold. This model consists of a two-dimensional square-shaped box filled with particles represented as point objects in continuous motion. The following assumptions are also assumed: 1) the particles are randomly distributed over the box initially; 2) all particles have the same absolute value of velocity; and 3) the initial headings of the particles are randomly distributed. Each particle updates its heading using the average of its own heading and the headings of all other particles within a radius r, which is called the *nearest neighbor rule*. Included in this model is a random noise (which may be viewed as measurement noise or actuation noise) with a uniform probability distribution on the interval $[-\eta, \eta]$. The result of [6] is to demonstrate using simulations that a phase transition occurs when the noise level crosses a threshold which depends on the particle density. Below the threshold, all particles tend to align their headings along some direction, and above the threshold, the particles move towards different directions. Czirók *et al* [7] presented a one-dimensional model which also exhibits phase transition for a group of mobile particles. These two models are discrete-time continuous-state models.

Schweitzer *et al* [8] studied the spatial-temporal evolution of the population of two species, where the update scheme depends nonlinearly on the local frequency of species. Depending on the probability of transition from one species to the other, the system evolves to either extinction of one species (agreement) or non-stationary co-existence or random co-existence (disagreement).

Problems that are closely related to phase transitions include, but not limited to, flocking / swarming / formation / consensus / agreement problems. Though the interest and focus of these problems are sometimes independent of the phase transition study, these problems typically exhibit phase transitions when parameters, conditions, or structures change. These problems and the phase transition problems may also share some common techniques in order to establish stability / instability over similar underlying models, such as common Lyapunov function techniques, graph theoretic techniques, and stochastic dynamical systems techniques. More specifically, the phase transitions occurring in flocking may be classified into two categories: angular phase transitions that leads to alignment (see e.g. [6]), and spatial self-organization in which multiple agents tend to form special patterns or structures in space, such as lattice type structures. Examples of the latter category include [9]–[11]. In [9], Mogilner and Edelstein-Keshet investigated swarming in which the dynamical objects interacts depending on angular orientations and spatial positions, and a phase transition is observed. In [10], Levine *et al* presented a simple model to study spatial self-organization in flocking showing that all the agents tend to localize in a special pattern in one- and in two-dimensions with all-to-all communication. We refer to [4], [11]–[14] for a recent study of phase transitions and the consensus / agreement problems over networks. Olfati-Saber [4] studied the consensus problem using a random rewiring algorithm (see also [15]) to connect nodes, and showed that the Laplacian spectrum of this network may undergo a dramatic shift, which is referred to as spectral phase transition and leads to extremely fast convergence to the consensus value. In [16], Hatano and Mesbahi established agreement of multiple agents over a network that forms an Erdös random graph process, in which each agent updates its state linearly according to the perfect state information of its nearest neighbors. Hatano and Mesbahi studied another facet of the distributed agreement problem in the stochastic setting in [14], namely the agreement over noisy network that forms a Poisson random graph.

Jadbabaie *et al* [17] provided a rigorous proof for the alignment of moving particles under the nearest neighbor rule without measurement noise or actuation noise. Different from the nonlinear switching model used in [6], the model in [17] is a switched linear model. Furthermore, this model also assumes that over every finite period of time the particles are *jointly connected* for the length of the entire interval. Due to the noiseless assumption made in [17], the phase transitions observed in [6] will not occur. Under these assumptions, Jadbabaie *et al* proved that the nearest neighbor rule leads to alignment of all particles. One may be interested in finding Lyapunov functions (preferably quadratic) to show the convergence or alignment (see [12], [13] for convergence proofs based on common Lyapunov functions for models different from [17]). However, [6] showed that a common quadratic Lyapunov function does not exist for this switched linear model. On the other hand, a non-quadratic Lyapunov function can be constructed to prove the convergence, as suggested by Megretski [18] and later independently found by Moreau [19]. See also [20], [21] for extension of [17].

In this paper, we propose a discrete-time discrete-state model in which a group of agents form either a fixed connected graph or a random graph process, and each agent (node) updates its value according to its previous value and the noisy measurements of the neighboring agent values. We prove that, when the noise level crosses some threshold from above, the system exhibits spontaneous symmetry breaking. We may view that the high noise level corresponds to high temperature (or strong thermal agitation), where the molecules exhibit disorder and symmetry; and the low noise level corresponds to low temperature, where the molecules exhibit order and asymmetry.

We emphasize that the proposed model is rather simple and hence admits complete mathematical analysis of the phase transition behavior. First, the phase transition in a fixed connected graph presented in this paper is simpler than the phase transition in the Ising model (as one indicator of the simplicity, the Ising model needs dimension two or higher to generate phase transition, whereas our phase transition occurs for any dimension; in addition, Ising models of dimension higher than two involves intractable computation complexity when attempting to solve the value for each node under the temperature constraint, more precisely, such a problem is NP-complete [22]). To the best of our knowledge, this model is one of the simplest that exhibits phase transition in a fixed graph, and is mathematically provable to generate phase transition. Note that many other phase transitions elude rigorous mathematical analysis due to their complexity [3], [4], [6], [7], [22]. Second, the phase transition on a random graph is also simpler than the phase transition on a random graph observed in [2]. Compared with the models in [6] and [7], our models have discrete-states and do not allow the mobility of agents, which greatly simplifies the systems dynamics and allows rigorous proofs of the sharp phase transition behavior. The simplicity of our phase transitions may help us to identify the essence of general phase transition phenomena.

Our study also sheds light on the research on the consensus problems, cooperation of multiple-agent systems, and collective behavior of complex systems, all under communication constraints. Hence, this study fits into the general framework of investigating the interactions between control/dynamical systems and information; see e.g. [23]–[29] and references therein. More specifically, we may interpret our phase transition in the consensus problem scenario, where the disagreement due to unreliable communication is replaced by agreement when the communication quality improves to a certain level. In other words, we characterize the significance of information in reaching agreement. However, unlike the average-consensus problem (cf. [13]) with the properties that, 1) there exists an invariant quantity during the evolution, and 2) the limiting behavior reaches the average of the initial states of the system, we reach a consensus without these properties when the noise level is low. This is because the presence of noise prevents the conserving of the sum of the node values during the evolution. The study of entropy flows (or information flow) [23], [30] may help identify an invariant quantity of the system. We remark that a more thorough study of the consensus problem raised in this paper is beyond the scope of this paper and will be pursued elsewhere.

Organization: In Section 2 we introduce the models. In Section 3 we state our main results and provide the proofs. In Section 4 we present numerical examples. Finally we conclude the paper and discuss future research directions.

II. Models on the graphs

This section introduces some of the terms that are frequently used in this paper as well as the two models to be investigated. We focus only on undirected graphs.

A. Graphs and random graph processes

A *graph* $G := (V, E)$ consists of a set $V := \{1, 2, ..., N\}$ of elements called vertices or nodes, and a set E of node pairs called edges, with $E \subseteq E_c := \{(i, j)|i, j \in V\}$. Such a graph is *simple* if it has no self loops, i.e. $(i, j) \notin E$ if $i = j$. We consider simple graphs only. A graph G is *connected* if it has a path between each pair of distinct nodes i and j, where by a *path* between nodes i and j we mean a sequence of distinct edges of G of the form $(i, k_1), (k_1, k_2), \ldots, (k_m, j) \in E$. Radius r from node i to node j means that the minimum path length, i.e., the minimum number of edges connecting i to j, is equal to r.

A *fixed* graph G has a node set V and an edge set E that consists of deterministic edges, that is, the elements of E are deterministic and do not change dynamically with time.

A *random* graph G consists of a node set V and an edge set $E := E(\omega)$, where $\omega \in \Omega$, $(\Omega, \mathcal{F}, P)$ forms a probability space. Here Ω is the set of all possible graphs of total number of n, where

$$n := 2^{\frac{N(N-1)}{2}}; \tag{1}$$

$\mathcal{F}$ is the power set of Ω; and P is a probability measure that assigns a probability to every $\omega \in \Omega$. In this paper, we focus on the well-known Erdös random graphs [5], namely, it holds that

$$P(\omega) = \frac{1}{n}. \tag{2}$$

In other words, we can view each $E(\omega)$ as a result of $N(N-1)/2$ independent tosses of fair coins, where a head corresponds to switching on the associated edge and a tail corresponds to switching off the associated edge. Notice that the introduction of randomness to a graph implies that, all results in random graph theory hold asymptotically and in a probability sense, such as "hold with probability one".

A *random graph process* is a stochastic process that describes a random graph evolving with time. In other words, it is a sequence $\{G(k)\}_{k=0}^{\infty}$ of random graphs (defined on a common probability space $(\Omega, \mathcal{F}, P)$) where k is interpreted as the time index (cf. [5]). Thus, for a random graph process, the edge set changes with k, and we denote the edge set at time k as

$E(k)$. In this paper, we assume that the edge formation at time k is independent of that at time l, if $k \neq l$.

The *neighborhood* $N_i(k)$ of the ith node at time k is a set consisting of all nodes within radius 1, including the ith node itself. The value that a node assumes is its *node value*. The *valence* or *degree* of the ith node is $(|N_i(k)| - 1)$, where $|N_i(k)|$ denotes the number of elements in $N_i(k)$. The *adjacency matrix* of $G(k)$ is an $N \times N$ matrix whose (i, j)th entry is 1 if the node pair $(i, j) \in E(k)$ and 0 otherwise. Note that the graphs can model lattice systems with any dimension.

B. System on a graph

A *system on a graph* consists of a graph, fixed or forming a random process, an initial condition that assigns each node a node value, and an update rule of the node values. In this paper, we assume that each node can take value either $+1$ or -1, and the *update rule* for the ith node at the $(k+1)$st instant is given by

$$x_i(k+1) = \text{sign}\left[v_i(k) + \xi_i(k)\right], \tag{3}$$

where $\xi_i(k)$ is the *noise* random variable, uniformly distributed in interval $[-\eta, \eta]$ and independent across time and space and independent of the initial condition $x(0)$, and

$$v_i(k) := \frac{\sum_{j \in N_i(k)} x_j(k)}{|N_i(k)|}; \tag{4}$$

that is, $v_i(k)$ is the average of the node values in the neighborhood $N_i(k)$. Here η is called the *noise level*. This update rule resembles the one in [7], with their antisymmetric function being replaced by a sign function. It may also be viewed as a specific update rule for a Hopfield neuron whose connections with others are noisy.

The *state of the system* at time instant k, denoted $x(k)$, is the collection of all node values $(x_1(k), \cdots, x_N(k))$. The *state sum* at time instant k, denoted $S(k)$, is defined as

$$S(k) := \sum_{i=1}^{N} x_i(k). \tag{5}$$

With a slight abuse of notation, we represent the state with all +1s and all -1s as $+N$ and $-N$, respectively. We call a state *transient* if this state reappears with probability strictly less than one. We call a state *recurrent* if this state reappears with probability one. We call a state X *absorbing* if the one-step transition probability from X to X is 1.

C. Model with a fixed graph

The first model considered is a system on a fixed graph. In this model, the node connections or the edges remain unchanged throughout. Hence, every node has a fixed neighborhood at all time, and the degree of each node as well as the adjacency matrix are constant. The node value gets updated according to the update rule (3). We will assume that the fixed graph is connected. An example of such a fixed graph model is a communication network with fixed nodes and fixed but noisy channels. Another example is a Hopfield network with fixed neurons and fixed but noisy channels connecting them.

D. Model with a random graph process

The second model considered is a system on a graph forming a random process. In this model, the node connections, namely the edges of the random graph, change dynamically throughout, and the edge formations at time k are random according to distribution $P(k)$. Hence every node may have different neighborhoods at different times, and the adjacency matrix and degrees change with time. The node value gets updated also according to the update rule (3). An example of this model could be an ad-hoc sensor network in which the communication links between the sensors are changing. Another example is an erasure network in which the communication channels are noisy and erasing with some probability, see for example [31].

In both models, the system state can take $2J$ values, where

$$J := 2^{N-1} \tag{6}$$

and the state sum takes values in the set $\mathcal{N} := \{-N, -N+2, \cdots, N-2, N\}$, where $N \geq 2$ is the total number of nodes (agents). Note that $|\mathcal{N}| = N+1 \geq 3$. Both models also form Markov chains, since the next state does not depend on previous state if the current state is given.

We use $\xi(k)$ to represent $(\xi_1(k), \cdots, \xi_N(k))$, ξ^k to represent $(\xi(0), \cdots, \xi(k))$, G^k to represent $(G(0), \cdots, G(k))$, and $x(k)$ to represent $(x_1(k), \cdots, x_N(k))$.

III. Main results and proofs

Our main result states that, *for a system on a fixed connected graph or on a graph forming a random process, there is a provable sharp phase transition when the noise level crosses some threshold*. Here phase transition is used in the sense that the symmetry exhibited at high noise level is broken suddenly when the noise level crosses the threshold from above, or equivalently the disagreement (or disorder) of the nodes at high noise level becomes agreement (or order) below the threshold. In what follows, we first discuss the case in which the graph has a fixed structure, and then the case in which the graph forms a random process.

A. Model with a fixed graph

Proposition 1. *For any given fixed connected graph, let D be the maximum number of nodes in one neighborhood.*

i) If the noise level is such that $\eta \in (1-2/D, 1]$, then the system will converge to agreement, *namely all nodes will converge to either all $+1$s or all -1s.*

ii) If the noise level is such that $\eta > 1$, then $\mathbf{E}_{S(0),\xi^{k-1}}S(k)$ tends to zero as k goes to infinity, i.e., the system will converge to disagreement *in which approximately half of the nodes are $+1$s and the other half are -1s.*

Remark 1. Notice that $(1-2/D)$ is guaranteed to be nonnegative for any connected graph with more than one node, since $D \geq 2$. Note also that if $\eta < (1-2/D)$, the system does not necessarily converge to states $\pm N$. To see this, simply consider a one-dimensional cellular automaton with N nodes forming a circle. The neighborhood of a node is defined as one node to the left, one node to the right, and itself. Therefore $D = 3$, and if $\eta < 1/3$, the update rule becomes a majority voting rule. Then the initial condition $x(0)$ of the system with alternate $+1$s and -1s will lead to constant oscillations between $x(0)$ and a left cyclic shift of $x(0)$, i.e., it will not reach agreement if $\eta < 1/3$. However, this does not mean that in general our condition $1 \geq \eta > (1-2/D)$ is a necessary condition for agreement; a sufficient and necessary condition is under current investigation. Attractors like this $x(0)$ may be viewed as local attractors (whereas $\pm N$ may be viewed as global attractors) which can be eliminated by considering a *randomized* graph, see the next subsection.

The proof of Proposition 1 needs the following lemmas.

Lemma 1. *For any given fixed connected graph, if $\eta \in (1-2/D, 1]$, then the states $\pm N$ are absorbing, and all other states are transient.*

Lemma 2. *For any given fixed connected graph, if $\eta > 1$, then the states form an ergodic Markov chain with a unique steady-state distribution for any initial condition $x(0)$.*

Proof of Lemma 1: Notice that at states $\pm N$, the noise is not strong enough to flip any node value. Thus, $\pm N$ are absorbing. On the other hand, all other states are neither absorbing nor recurrent. To see this, let $M \neq \pm N$ be any state. In such a case, M contains a mixture of $+1$s and -1s. Due to connectivity of the graph, we can always find a node i with node value $x_i(k) = -1$ whose neighborhood $N_i(k)$ (including $x_i(k)$ itself) contains both $+1$s and -1s.

Then for such $x_i(k)$, it holds that

$$|v_i(k)| \leq \frac{D-2}{D}, \tag{7}$$

with equality if only one node in $N_i(k)$ has a different sign than all other nodes and if $N_i(k)$ contains D nodes. Hence a noise larger than $(D-2)/D$ flips $x_i(k)$. Precisely,

$$\begin{aligned} & \Pr[x_i(k+1)=+1|x_i(k)=-1] \\ = & \Pr\left[v_i(k)+\xi_i(k)>0|x_i(k)=-1\right] \\ \geq & \Pr\left[\xi_i(k)>\frac{D-2}{D}\middle| x_i(k)=-1\right] \\ = & \frac{1}{2}\left(1-\frac{D-2}{D\eta}\right)>0. \end{aligned} \tag{8}$$

Note that the conditioning is removed due to the independence assumptions on noise. Thus, for state M, the probability that only x_i flips and no other node changes its value is non-zero. This follows that, with a positive probability the state sum for M will be increased by 2. Similarly, with a positive probability M can decrease by 2. Since $M \neq \pm N$ is an arbitrary state, by induction, the probability of transition (in possibly multiple steps) from M to $\pm N$ is nonzero. So M is transient. ∎

Proof of Lemma 2: It is sufficient to prove that the state forms an irreducible and aperiodic Markov chain.

To see the irreducibility, note that if $\eta > 1$, $M \neq \pm N$ can jump to any other states with a positive probability, similar to Lemma 1. Additionally, $\pm N$ can also jump to any other states with a positive probability. For state $+N$, it holds that

$$\begin{aligned} & \Pr[x_i(k+1)=-1|x_l(k)=+1, l=1,\cdots,N] \\ = & \Pr\left[\frac{\sum_{j\in N_i(k)}+1}{|N_i(k)|}+\xi_i(k)<0\middle| x_l(k)=+1, l=1,\cdots,N\right] \\ = & \Pr[\xi_i(k)<-1|x_l(k)=+1, l=1,\cdots,N] \\ = & \frac{1}{2\eta}(\eta-1)>0, \end{aligned} \tag{9}$$

so any node can flip its value with a positive probability. Similar result holds for state $-N$. Then this Markov chain is irreducible.

To see the aperiodicity, let us use $-x_i$ to denote the flipped x_i. The state transition loop from $(x_1(k), x_2(k), *)$ to $(-x_1(k), -x_2(k), *)$ to $(-x_1(k), x_2(k), *)$ and back to $(x_1(k), x_2(k), *)$ has period 3, where $*$ is any fixed configuration for $(x_3(k), \cdots, x_N(k))$. However, the state transition loop from $(x_1(k), \Delta)$ to $(-x_1(k), \Delta)$ and back to $(x_1(k), \Delta)$ has period 2, where Δ is any fixed configuration for $(x_2(k), \cdots, x_N(k))$. Note that such loops occur with positive probabilities. Then the Markov chain is aperiodic. ∎

Proof of Proposition 1: If $\eta \in (1-2/D, 1]$, from Lemma 1, the associated Markov chain will converge to either $+N$ or $-N$ with probability 1, namely agreement. If $\eta > 1$, from Lemma 2 we know that the associated Markov chain is ergodic, and notice that the Markov chain has a symmetric structure for states x and $-x$. Then $\pi(x) = \pi(-x)$ (rigorous proof is included in Appendix), where $\pi(x)$ is the stationary probability of state x. Hence the expectation of state sum is

$$\mathbf{E}_{S\sim\pi} S = \sum_{x} \left(\pi(x) \sum_{i=1}^{N} x_i \right) = 0. \tag{10}$$

Therefore, $\mathbf{E}S(k)$ converges to zero, and asymptotically the numbers of $+1$s and -1s will asymptotically become equal. ∎

B. Model with a random graph process

For an Erdös random graph, we assume that the edge connections are randomly and independently changing from time to time. The randomization of the connections symmetrizes the system behavior and leads to agreement for an arbitrarily small but positive noise level.

Proposition 2. *Consider an Erdös random graph process.*

i) If the noise level is such that $0 < \eta \leq 1$, then the system will converge to agreement, *namely the state will converge to $+N$ or $-N$.*

ii) If the noise level is such that $\eta > 1$, then $\mathbf{E}S(k)$ exponentially converges to zero with decay exponent $\ln \eta$ as k goes to infinity, i.e., the system will exponentially converge to disagreement *in which about half of the node values are $+1$s and the other half are -1s.*

The proof of this proposition needs the following lemmas. We remark that it is straightforward to generalize the lemmas to a binomial random graph, in which the probability of forming an edge is changed from 0.5 to an arbitrary $p \in (0, 1)$.

Lemma 3. *For any Erdös random graph process, if $0 < \eta \leq 1$, then $\pm N$ are absorbing, and all other states are transient.*

Lemma 4. *For any Erdös random graph process, if $\eta > 1$, then it holds that $\mathbf{E}S(k)$ exponentially tends to zero as k goes to infinity.*

Proof of Lemma 3: If $0 < \eta \leq 1$, it is easy to see that $\pm N$ are absorbing. For any state $M \neq \pm N$, it holds that M must be a mixture of both $+1$s and -1s. Hence we can find i and j in V such that $x_i(k) = -1$ and $x_j(k) = +1$. Since each one of the n graphs (recall (1)) has a positive probability, the probability that x_i is connected to x_j only is positive. Then in this case, the corresponding $v_i(k)$ is 0 and hence an arbitrarily small but positive noise may flip x_i with a positive probability. In addition, all nodes other than x_i have a positive probability to keep their previous values, thus with a positive probability, the state sum for M can be increased by 2. Therefore any $M \neq \pm N$ are transient. ∎

Proof of Lemma 4: For any Erdös random graph, if $\eta > 1$, then no state is absorbing, since with a positive probability the noise can flip any node value in any configuration. Therefore, with a nonzero probability the state of the system can jump to any other states.

Now let us analyze the evolution of $\mathbf{E}S(k)$. Fix the time to be k. Assume $x(k)$ is given. Then for each i, $x_i(k+1)$ is given by (3). The randomness in $x_i(k+1)$ is due to the noise $\xi_i(k)$ and the graph $G(k)$. It holds that

$$\begin{aligned}
& \mathbf{E}[x_i(k+1)|x(k)] \\
= \; & \mathbf{E}\,\text{sign}[v_i(k)+\xi_i(k)|x(k)] \\
= \; & \text{Pr}[v_i(k)+\xi_i(k)>0|x(k)]\times(+1) \\
& +\text{Pr}[v_i(k)+\xi_i(k)<0|x(k)]\times(-1) \\
= \; & \text{Pr}[\xi_i(k)>-v_i(k)|x(k)]-\text{Pr}[\xi_i(k)<-v_i(k)|x(k)] \\
= \; & \sum_{v_i(k)}\text{Pr}[\xi_i(k)>-v_i(k)]\text{Pr}[v_i(k)|x(k)] \\
& -\sum_{v_i(k)}\text{Pr}[\xi_i(k)<-v_i(k)]\text{Pr}[v_i(k)|x(k)] \\
= \; & \sum_{v_i(k)}\left[\frac{\eta+v_i(k)}{2\eta}-\frac{\eta-v_i(k)}{2\eta}\right]\text{Pr}[v_i(k)|x(k)] \\
= \; & \sum_{v_i(k)}\frac{v_i(k)}{\eta}\text{Pr}[v_i(k)|x(k)] \\
= \; & \frac{1}{\eta}\mathbf{E}[v_i(k)|x(k)].
\end{aligned} \tag{11}$$

Then we compute $\mathbf{E}(v_i(k)|x(k))$. Since conditioned on $x(k)$, the randomness in $v_i(k)$ comes from $G(k)$ only, this expectation boils down to the expectation of the average of node values in a neighborhood, averaged over all possible n graph structures. Let us count in the n graph structures the number of different neighborhood types containing node i. Among those neighborhoods containing node i, there are

$$\bar{m} := 2^{(N-1)(N-2)/2}\times\binom{N-1}{m} \tag{12}$$

types of neighborhoods for which $|N_i(k)| = (m+1)$ where $m = 0, 1, \cdots, N-1$. To see this, simply notice that the graph formed by nodes other than i can have any edge formation and hence the number of types of $2^{(N-1)(N-2)/2}$, and that node i needs to select m out of the other $(N-1)$ nodes in order to have $|N_i(k)| = (m+1)$.

Therefore,

$$\begin{aligned} & \mathbf{E}[v_i(k)|x(k)] \\ = & \sum_{G(k)} [v_i(k)|x(k), G(k)] \Pr[G(k)|x(k)] \\ \stackrel{(a)}{=} & \frac{1}{n} \sum_{G(k)} \left[\frac{\sum_{j \in N_i(k)} x_j(k)}{|N_i(k)|} \middle| x(k), G(k) \right] \\ = & \frac{1}{n} \sum_{G(k)} \left[\frac{x_i(k)}{|N_i(k)|} \middle| x(k), G(k) \right] + \\ & \quad \frac{1}{n} \sum_{G(k)} \left[\frac{\sum_{j \in N_i(k), j \neq i} x_j(k)}{|N_i(k)|} \middle| x(k), G(k) \right], \end{aligned} \tag{13}$$

where (a) is due to (2), the independence assumptions, and definition of $v_i(k)$. Now first note that

$$\sum_{G(k)} \left[\frac{x_i(k)}{|N_i(k)|} \middle| x(k), G(k) \right] = \sum_{m=0}^{N-1} \left[\frac{x_i(k)}{m+1} \bar{m} \right]. \tag{14}$$

Then note that in the summation

$$\sum_{G(k)} \left[\frac{\sum_{j \in N_i(k), j \neq i} x_j(k)}{|N_i(k)|} \middle| x(k), G(k) \right], \tag{15}$$

each node $j \neq i$ will be counted for $\bar{m} \times \frac{m}{N-1}$ times for those neighborhood types such that $j \in N_i(k)$ and $|N_i(k)| = (m+1)$, so it holds that

$$\sum_{G(k)} \left[\frac{\sum_{j \in N_i(k), j \neq i} x_j(k)}{|N_i(k)|} \middle| x(k), G(k) \right] = \sum_{j \neq i} \sum_{m=0}^{N-1} \left[\frac{x_j(k)}{m+1} \bar{m} \frac{m}{N-1} \right]. \tag{16}$$

Thus, we have

$$\mathbf{E}_{G(k)} v_i(k) = c_1 x_i(k) + \sum_{j \neq i} c_2 x_j(k), \tag{17}$$

where

$$\begin{aligned} c_1 & := \frac{2^{(N-1)(N-2)/2}}{n} \sum_{m=0}^{N-1} \binom{N-1}{m} \times \frac{1}{m+1}, \\ c_2 & = \frac{2^{(N-1)(N-2)/2}}{n(N-1)} \sum_{m=1}^{N-1} \binom{N-1}{m} \times \frac{m}{m+1}. \end{aligned} \tag{18}$$

This yields that, using (11),

$$\mathbf{E}[x_i(k+1)|x(k)] = \frac{1}{\eta} \left[c_1 x_i(k) + c_2 \sum_{j \neq i} x_j(k) \right], \tag{19}$$

and hence

$$\begin{aligned}
& \mathbf{E}[S(k+1)|x(k)] \\
= & \sum_{i=1}^{N} \mathbf{E}[x_i(k+1)|x(k)] \\
= & \frac{1}{\eta}\left[c_1 S(k) + c_2(N-1)S(k)\right] \\
= & \frac{1}{\eta} S(k) \frac{1}{2^{N-1}} \sum_{m=0}^{N-1} \binom{N-1}{m} \\
= & \frac{1}{\eta} S(k).
\end{aligned} \tag{20}$$

Therefore, to find the expectation over time, we deduce

$$\begin{aligned}
& \mathbf{E}(S(k+1)) \\
= & \mathbf{E}[\mathbf{E}(S(k+1)|x(k))] \\
= & \frac{1}{\eta}\mathbf{E}S(k).
\end{aligned} \tag{21}$$

Since $\eta > 1$, the above recursion converges to zero exponentially, and the decay exponent is

$$-\frac{1}{k}\ln\frac{\mathbf{E}S(k)}{\mathbf{E}S(0)} = -\ln\frac{1}{\eta} = \ln\eta. \tag{22}$$

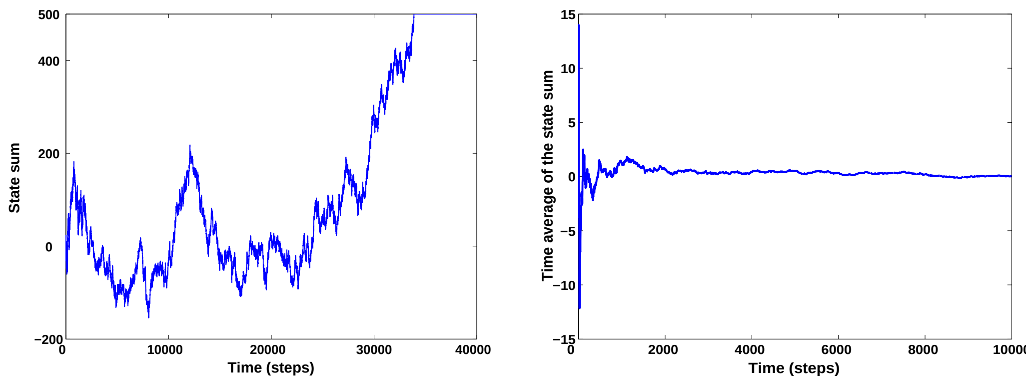
■

Proof of Proposition 2: If $0 < \eta \leq 1$, from Lemma 3, the system state sum will converge to the absorbing states with probability 1, namely agreement. If $\eta > 1$, from Lemma 4, the system state will converge to zero exponentially with probability 1. ■

IV. NUMERICAL RESULTS

A. Fixed graph case

Consider a fixed one-dimensional 500-agent system. The agents are listed along a circle and each agent has two neighbors. The initial value of every agent is arbitrarily assigned to be $+1$ or -1. The simulation results demonstrate the phase transitions, see Figure 1 (a) and (b). In Figure 1 (a), the vertical axis represents the state sum of the system, and the horizontal axis represents the simulation steps. Figure 1 (a) demonstrates that, when the noise level is such that $1/3 < \eta \leq 1$, then all node values converge to agreement of all $+1$s (or all -1s), that is, the state sum of the system is +500 (or -500). In Figure 1 (b), the vertical axis represents the time average of the state sum, and the horizontal axis is for the simulation steps. By ergodicity of the system, the time average should converge to the ensemble average of the state sum. Figure 1 (b) shows that, if the noise level is such that $\eta > 1$, then all node values converge to disagreement in which about half of the node values are $+1$s and the other half are -1s. Clearly, the noise level equal to 1 is the critical level of the phase transition.



(a) (b)

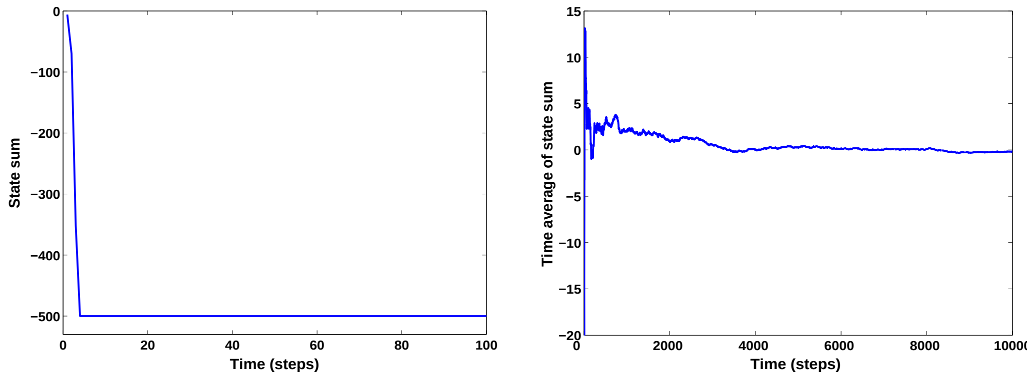
Fig. 1. Fixed graph simulation. (a) Noise level is 0.75, and the state sum converges to agreement of all $+1$s. (b) Noise level is 2, and the state sum converges to disagreement in which about half of the states are $+1$s and the other half are -1s.

B. Random graph process case

For the random graph process case, in our simulation we consider *binomial random graphs*. In a binomial random graph, each edge has a probability p to be formed at each time step and is independent of all other edges and other times. This means that to generate such a binomial random graph, we only need to generate at each step an adjacency matrix whose entries in the upper triangular part are independent and identically distributed. The initial value of every agent is randomly assigned to be $+1$ or -1 according to an arbitrary distribution. The simulation results are shown in Figure 2 (a) and (b), and are similar to the fixed connected graph case, except that in the random graph case, an arbitrarily small but positive noise level can lead to agreement.

We can also compute the decay exponent of $\mathbf{E}S(k)$ from the numerical results. Note that to obtain the probability mean $\mathbf{E}S(k)$ numerically, we can run many independent trials of the random process and take the average of the state sums across the trials. See Figure 3 for the simulated decay exponents (with different edge probability p) and the theoretic decay exponent $\ln \eta$, which are almost identical.

Notice that p does not play any role in the decay exponent. The role of p is reflected in other quantities, such as the stationary distribution. To see this, let us consider a two-node binomial graph, i.e. $N = 2$, and compute the stationary distribution as well as the decay exponent



(a) (b)

Fig. 2. (a) System is randomly connected, noise level is 0.005, $p = 0.1$, and all states converge to agreement of all -1s. (b) System is randomly connected, noise level is 2, $p = 0.2$, and the state converges to disagreement in which about half of the states are $+1$s and the other half are -1s.

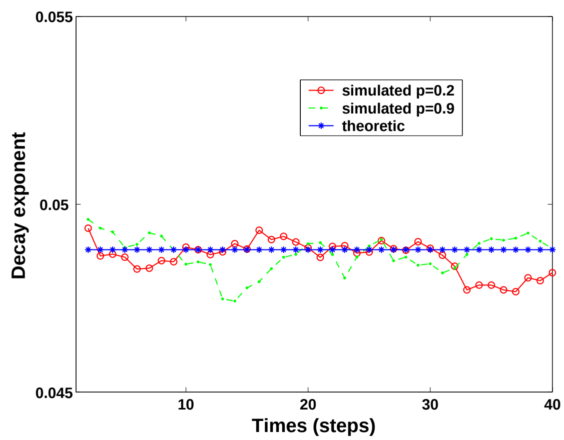


Fig. 3. The simulated decay exponents (averaged over 10,000 independent trials and $\eta = 1.05$) and the theoretic decay exponent.

analytically in a different way. We order the state values as $(+1,+1)'$, $(+1,-1)'$, $(-1,-1)'$, and $(-1,+1)'$. Based on this order, the transition probability matrix is

$$p = \left[\begin{array}{c|c|c|c} c & \frac{p}{4}+qb & a & \frac{p}{4}+qb \\ \hline b & \frac{p}{4}+qc & b & \frac{p}{4}+qa \\ \hline a & \frac{p}{4}+qb & c & \frac{p}{4}+qb \\ \hline b & \frac{p}{4}+qa & b & \frac{p}{4}+qc \end{array}\right] \tag{23}$$

and the stationary distribution is

$$p = \begin{bmatrix} \frac{1}{2}\frac{p+4qb}{p+4(1+q)b} \\ \frac{2b}{p+4(1+q)b} \\ \frac{1}{2}\frac{p+4qb}{p+4(1+q)b} \\ \frac{2b}{p+4(1+q)b} \end{bmatrix} \tag{24}$$

where $q := (1-p)$, and

$$\begin{aligned} a &:= \frac{(\eta-1)^2}{4\eta^2} \\ b &:= \frac{(\eta-1)(\eta+1)}{4\eta^2} \\ c &:= \frac{(\eta+1)^2}{4\eta^2}. \end{aligned} \tag{25}$$

Clearly, the edge probability p influences the stationary distribution. Now assume that the state is distributed according to $p_0 := [p_{+2}, p_{+1}, p_{-2}, p_{-1}]$, which has the expected state sum as $2(p_{+2} - p_{-2})$. Then the state at the next time will be distributed as $p \times p_0$, and the expected state sum will become $2(a-c)(p_{+2} - p_{-2}) = 2(p_{+2} - p_{-2})/\eta$. This verifies the independence of the decay rate on p.

V. Conclusions and future work

In this paper, we proposed simple dynamical systems models exhibiting sharp phase transitions, and presented complete, rigorous proofs of phase transition behavior, with thresholds found analytically. Our analysis also provided a characterization of how information (or noise) affects the collective behavior of multi-agent systems, which gives an analytic explanation to the intuition that, to reach consensus, high quality of communication is needed. These results hold for any dimension; in contrast, phase transitions in the well known Ising models do not occur for dimension one, and for dimension three or higher, Ising models are NP complete and intractable.

In particular, we have shown that for a fixed connected graph, if the noise level is greater than $(1 - 2/D)$ and less than 1, all the agents reach an agreement, i.e. the state sum of the system converges to $\pm N$, the only absorbing states of the system. For noise level larger than 1, the group of agents fail to reach any agreement; instead they reach “complete disagreement” or disorder. Thus, phase transition occurs at $\eta = 1$. For random graph processes, the system reaches agreement even for noise level smaller than $(1 - 2/D)$. This is because randomization is immune to the artifacts (or local attractors) for smaller noise which stops fixed graph from reaching any agreement. However, the tradeoff is that in randomization, the nodes’ neighbors are random, and therefore the neighbors may not be “geographically close”, which might not be feasible in practical situations.

Our study was concentrated on the leaderless case. The leader case is when there is a leader with a fixed value and it tries to convince all other agents to follow its value. Simulation obtained in this case suggested that a complete analysis is a bit involved especially in the high noise regime, which is subject to further research. Another direction could be to obtain a suitable Lyapunov function for the models. One advantage of doing so is that the Lyapunov function based approach may be extended to rather general nonlinear systems, as suggested by [18], [19]. The Lyapunov function is preferably a quadratic one, leading to stability in the mean-square sense, which is stronger than the stability in the mean sense obtained in this paper. The applications of our approach and results are also subject to future research, including the extension of our approach to more realistic models; note that our models in this paper are simple and not realistic enough, though the simplicity helped us to completely characterize the phase transition. We will also explore the connections of our model to relevant models, such as the Ising models, Hopfield networks, cellular automata, other random graphs, etc. Finally, we remark that the approach and results developed in this paper may be found useful to study more general dynamical systems under communication constraints, such as cooperation with limited communication, complex systems in the presence of noise, etc. The study of such problems would help establish insights on how information (or limited information) interacts with system dynamics to generate various types of interesting system behavior.

APPENDIX

We prove that $\pi(x) = \pi(-x)$ for any x in four steps.

Step 1: Establish a one-to-one mapping between the $2J$ possible values (see (6)) that the state of the system can take and integers $\pm 1, \pm 2, \cdots, \pm J$, such that if state x is mapped to $+j$, then state $-x$ is mapped to $-j$. Now aggregate the states as follows. Let $\bar{j} := (j, -j)$ for any $j = 1, \cdots, J$. Then we induce from the Markov process $\{x(k)\}_{k=0}^{\infty}$ another Markov process $\{\bar{x}(k)\}_{k=0}^{\infty}$, where the latter is defined on the induced state space (i.e. the set formed by all $\bar{j}$s). Note that it is straightforward to verify that $\{\bar{x}_k\}_{k=0}^{\infty}$ forms a Markov process on the induced state space, and this Markov process is ergodic, since the underline process is ergodic.

Step 2: Denote the transition probability matrix for process $\{\bar{x}(k)\}_{k=0}^{\infty}$ as $\bar{p}$, and the corresponding stationary distribution vector as $\bar{\pi} := [\bar{\pi}(\bar{1}), \bar{\pi}(\bar{2}), ..., \bar{\pi}(\bar{J})]'$. Then it holds that $\bar{\pi} = \bar{p}\bar{\pi}$. By ergodicity, $\bar{\pi}$ is non-zero and unique (i.e., the matrix $(I - \bar{p})$ must be rank deficient).

Step 3: For the Markov process $\{x(k)\}_{k=0}^{\infty}$, denote the stationary distribution vector as $\pi := [\pi_1', \pi_2']'$, where $\pi_1' = [\pi(+1), \pi(+2), ..., \pi(+J)]'$ and $\pi_2' = [\pi(-1), \pi(-2), ..., \pi(-J)]'$. It can be verified that, by the symmetry that the state transition $i \to j$ has the same probability as the state transition $(-i) \to (-j)$, the transition probability matrix has the following particular form:

$$p := \begin{bmatrix} A & B \\ B & A \end{bmatrix}. \tag{26}$$

Step 4: By ergodicity of $\{x(k)\}_{k=0}^{\infty}$, it holds that

$$\pi = p\pi \tag{27}$$

or equivalently,

$$\begin{aligned} \pi_1 &= A\pi_1 + B\pi_2 \\ \pi_2 &= B\pi_1 + A\pi_2. \end{aligned} \tag{28}$$

However, it can be easily seen that $\bar{p} = A + B$. Notice that

$$\pi_1 = \pi_2 = \bar{\pi} \tag{29}$$

is a solution to (28), i.e., $\pi_0 := [\bar{\pi}', \bar{\pi}']'$ solves (27). Since $\bar{\pi}$ is non-zero, π_0 is a non-zero solution to (27). By ergodicity, the non-zero solution is unique, and hence π_0 must be the solution to (27), which follows that $\pi(j) = \pi(-j)$ for any j or $\pi(x) = \pi(-x)$ for any x.

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